Solutions
Math 220
HW # 8
November 13, 2018

Exercise 1. *Let A and B be sets inside some universal set. Prove DeMorgan's Law*

$$(A \cap B)^c = A^c \cup B^c.$$

Proof.

$$\begin{array}{rl} & x \in (A \cap B)^c \\ \iff & x \notin A \cap B \\ \iff & \text{it is not true that } x \in A \text{ and } x \in B \\ \iff & x \notin A \text{ or } x \notin B \\ \iff & x \in A^c \text{ or } x \in B^c \\ \iff & x \in A^c \cup B^c \end{array}$$

□

Exercise 2. *Let $A_1, A_2, ..., A_n$ be a collection of subsets of some universal set. Prove the extended version of DeMorgan's Law*

$$(A_1 \cap A_2 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c.$$

Proof. We prove this by induction.

Base Case $(n = 2)$

The base case is the standard version of DeMorgan's Law from the first exercise.

Ind. Hyp. Assume for some $k \geq 2$

$$(A_1 \cap A_2 \cap \cdots \cap A_k)^c = A_1^c \cup A_2^c \cup \cdots \cup A_k^c.$$

Ind. Step We wish to prove

$$(A_1 \cap A_2 \cap \cdots \cap A_{k+1})^c = A_1^c \cup A_2^c \cup \cdots \cup A_{k+1}^c.$$

Observe

$$\begin{aligned} (A_1 \cap A_2 \cap \cdots \cap A_{k+1})^c &= (A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1})^c \\ &= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1})^c \\ &= (A_1 \cap A_2 \cap \cdots \cap A_k)^c \cup A_{k+1}^c \\ &= (A_1^c \cup A_2^c \cup \cdots \cup A_k^c) \cup A_{k+1}^c \end{aligned}$$

as desired.

□

Exercise 3. *Let A and B be sets. Prove:*

$$\textit{If } A \subset B, \textit{ then } A \cap B^c = \varnothing.$$

Proof. Suppose $A \cap B^c \neq \varnothing$. Then:

$$\begin{aligned} & x \in A \cap B^c \\ \Longrightarrow \quad & x \in A \text{ and } x \in B^c \\ \Longrightarrow \quad & x \in A \text{ and } x \notin B \end{aligned}$$

However, since $A \subset B$, if $x \in A$, then $x \in B$. So we have a contradiction. Therefore $A \cap B^c = \varnothing$. □

Exercise 4. *Prove for sets A, B, C*

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Proof.

$$\begin{aligned} & (x, y) \in A \times (B \cap C) \\ \Longleftrightarrow \quad & x \in A \text{ and } y \in B \cap C \\ \Longleftrightarrow \quad & x \in A \text{ and } (y \in B \text{ and } y \in C) \\ \Longleftrightarrow \quad & (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ \Longleftrightarrow \quad & (x, y) \in A \times B \text{ and } (x, y) \in A \times C \\ \Longleftrightarrow \quad & (x, y) \in (A \times B) \cap (A \times C) \end{aligned}$$

□

Exercise 5. *Prove or disprove: For sets A, B, C*

$$(A \cap B) \cup C = A \cap (B \cup C).$$

Proof. This is not true. Let $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Then $(A \cap B) \cup C = \{1, 2\}$, but $A \cap (B \cup C) = \{1\}$. Therefore this is a counterexample to the equality. □

Exercise 6. *Fill in the blanks with the properties (from the class handout) used to prove this result*

$$\begin{aligned} (A \cup B) \setminus (C \setminus A) &= (A \cup B) \cap (C \setminus A)^c && \textit{by } \underline{(a)} \\ &= (A \cup B) \cap (C \cap A^c)^c && \textit{by } \underline{(b)} \\ &= (A \cup B) \cap (A^c \cap C)^c && \textit{by } \underline{(c)} \\ &= (A \cup B) \cap ((A^c)^c \cup C^c) && \textit{by } \underline{(d)} \\ &= (A \cup B) \cap (A \cup C^c) && \textit{by } \underline{(e)} \\ &= A \cup (B \cap C^c) && \textit{by } \underline{(f)} \\ &= A \cup (B \setminus C) && \textit{by } \underline{(g)} \end{aligned}$$

Solution.

(a) Set Difference
(b) Set Difference
(c) Commutative
(d) DeMorgan's
(e) Double Complement
(f) Distributive
(g) Set Difference

□

Exercise 7. *Give an "algebraic proof" (as in the previous exercise) of the identity*

$$((A^c \cup B^c) \setminus A)^c = A.$$

Proof.

$$\begin{aligned} ((A^c \cup B^c) \setminus A)^c &= ((A^c \cup B^c) \cap A^c)^c && \text{by } \underline{\text{Set Difference}} \\ &= (A^c \cup B^c)^c \cup A && \text{by } \underline{\text{DeMorgan and Double Complement}} \\ &= (A \cap B) \cup A && \text{by } \underline{\text{DeMorgan and Double Complement}} \\ &= A && \text{by } \underline{\text{Absorption}} \end{aligned}$$

□

Exercise 8. *Let X be a set. Show that $\mathscr{P}(X)$ satisfied the definition to be a topology on the set X. (This is called the* discrete topology *on X.)*

Proof. We need to verify the three pieces:

(1) Both $\varnothing$ and X are subsets of X, so $\varnothing, X \in \mathscr{P}(X)$.
(2) Let $\{V_1, V_2, V_3, ...\}$ be a collection of subsets of X. If $x \in \bigcup_{i=1}^{\infty} V_i$, then $x \in V_k$ for at least one k. But since $V_k \subset X$, we have $x \in X$. Therefore $\bigcup_{i=1}^{\infty} V_i \subset X$, hence $\bigcup_{i=1}^{\infty} V_i \in \mathscr{P}(X)$.
(3) Let $\{V_1, V_2, ..., V_n\}$ be a collection of subsets of X. If $x \in \bigcap_{i=1}^{n} V_i$, then $x \in V_k$ for all k. But since $V_k \subset X$ for all k, we have $x \in X$. Therefore $\bigcap_{i=1}^{n} V_i \subset X$, hence $\bigcap_{i=1}^{\infty} V_i \in \mathscr{P}(X)$.

□

Exercise 9. *Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3, 4, 5, 6\}$. Define the function $f : X \to Y$ by*

$$f = \{(a, 3), (b, 5), (c, 2), (d, 3), (e, 6)\}.$$

(a) *What is the domain and co-domain of f?*
(b) *Give the values of $f(a)$, $f(b)$, and $f(e)$.*
(c) *What is the range of f?*

(*d*) *Is c the inverse image of* 2*?*

(*e*) *What is the inverse image of* 3*? the inverse image of* 4*?*

Solution.

(*a*) Domain of f is X, co-domain of f is Y.

(*b*) $f(a) = 3$, $f(b) = 5$, and $f(e) = 6$.

(*c*) The range of f is $f(X) = \{2, 3, 5, 6\}$.

(*d*) Yes, $(c, 2)$ is the only pair that 2 appears in, so c is the inverse image of 2.

(*e*) $f^{-1}(\{3\}) = \{a, d\}$. $f^{-1}(\{4\}) = \varnothing$.

□

Exercise 10. *Let $f : X \to Y$ be a function. Let A and B be subsets of X. Below are two statements, one of which is true and one of which is false. Decide which is true and which is false. Give a counterexample to verify the false one, and prove the true one.*

$$\begin{aligned} f(A \cap B) &= f(A) \cap f(B) \\ f(A \cup B) &= f(A) \cup f(B) \end{aligned}$$

Proof. $f(A \cap B) \neq f(A) \cap f(B)$.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Let $A = [-1, 0]$ and $B = [0, 1]$. Then $A \cap B = \{0\}$ so $f(A \cap B) = \{0\}$. However, $f(A) = f(B) = [0, 1]$, so $f(A) \cap f(B) = [0, 1] \neq \{0\}$.

To prove the other result, observe

$$\begin{aligned} & x \in f(A \cup B) \\ \iff & \exists y \in A \cup B, f(y) = x \\ \iff & \exists (y \in A \text{ or } y \in B), f(y) = x \\ \iff & \exists y \in A, f(y) = x \text{ or } \exists y \in B, f(y) = x \\ \iff & x \in f(A) \text{ or } x \in f(B) \\ \iff & x \in f(A) \cup f(B) \end{aligned}$$

□